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Strong Unicity and Lipschitz Conditions of Order $\frac{1}{2}$ for Monotone Approximation

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1. INTRODUCTION

Let π_n denote the set of real algebraic polynomials of degree n or less and let $\|\cdot\|$ be the uniform norm on $C[a, b]$, the set of continuous, real-valued functions defined on the interval $[a, b]$. The classical strong unicity theorem (see Cheney [1], p. 80) asserts that if p_f is the best uniform approximation to $f \in C[a, b]$ from π_n, then there is a constant $\gamma > 0$ such that

$$\|f - p\| \geqslant \|f - p_f\| + \gamma \|p - p_f\| \tag{1.1}$$

or, equivalently,

$$\|p - p_f\| \leqslant (1/\gamma)(\|f - p\| - \|f - p_f\|) \tag{1.2}$$

for all $p \in \pi_n$. In a recent paper, Y. Fletcher and J. A. Roulier [2] have shown that strong unicity can fail for monotone approximation in which the approximating space consists of the nondecreasing polynomials in π_n. In this paper, it is shown that an alternative concept which we call "strong unicity of order $\frac{1}{2}$" is valid for monotone approximation. The setting of this paper is somewhat more general than that of [2].

Let $1 \leqslant k_1 < k_2 < \cdots < k_l \leqslant n$ be l integers and $\epsilon_i = \pm 1$ $(i = 1,..., l)$ be l signs. In the monotone approximation problem considered by G. .G. Lorentz and K. L. Zeller [4] and R. A. Lorentz [6], the approximating space is

$$\begin{aligned} M_n &= M_n(k_1, ..., k_l; \epsilon_1, ..., \epsilon_l) \\ &= \{p \in \pi_n\colon \epsilon_i p^{(k_i)}(x) \geqslant 0 \text{ for } x \in [a, b],\ i = 1,..., l\}. \end{aligned}$$

The approximating space in [2] is $M_n(1; 1)$. If p_f is the best uniform approximation to $f \in C[a, b]$ from M_n, we say that p_f is *strongly unique of order* α $(0 < \alpha \leqslant 1)$ if for each $K > 0$ there is a constant $\gamma > 0$ such that

$$\|f - p\| \geqslant \|f - p_f\| + \gamma \|p - p_f\|^{1/\alpha} \tag{1.3}$$

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or, equivalently,

$$\| p - p_f \| \leqslant (1/\gamma)^\alpha (\| f - p \| - \| f - p_f \|)^\alpha \tag{1.4}$$

for all $p \in M_n$ with $\| p \| \leqslant K$. The principle result of this paper is that if n is large enough so that $\deg p_f \geqslant k_l$, then best monotone approximations are strongly unique of order $\frac{1}{2}$. The example of Fletcher and Roulier [2] shows that the order $\frac{1}{2}$ cannot in general be improved. In doing this, we shall need to extend a modified strong unicity result of Fletcher and Roulier [2] to the more general setting of approximation from $M_n(k_1 ,..., k_l ; \epsilon_1 ,..., \epsilon_l)$. This is accomplished with no change in their proof.

It is known that strong unicity and Lipschitz conditions for best approximation operators are related (see Cheney [1], p. 82). In this light, the best uniform approximation operator corresponding to $M_n(k_1 ,..., k_l ; \epsilon_1 ,..., \epsilon_l)$ is shown to satisfy a local Lipschitz condition of order $\frac{1}{2}$ on bounded subsets of $C[a, b]$ when $\deg p_f \geqslant k_l$. In addition, this operator is continuous even if $\deg p_f < k_l$.

2. Background and Notations

The results of this paper depend on a characterization theorem of G. G. Lorentz and K. L. Zeller [4] and a modification of a lemma of R. A. Lorentz [6] used in establishing uniqueness of best approximations from M_n. In this section, we state the pertinent results from these papers and introduce a seminorm and a norm on π_n to be used in subsequent analyses.

Let $f \in C[a, b]\backslash M_n$ and $p_f \in M_n$. Define

$$A = A(f, p_f) = \{x \in [a, b] : |f(x) - p_f(x)| = \| f - p_f \|\} \tag{2.1}$$

and

$$B_i = B_i(p_f) = \{y \in [a, b]: p_f^{(k_i)}(y) = 0\} \tag{2.2}$$

for $i = 1,..., l$. For $x \in A$, let

$$\sigma(x) = \operatorname{sgn}[f(x) - p_f(x)].$$

The characterization theorem of [4] follows.

LEMMA 2.1. *Let $f \in C[a, b]\backslash M_n$ and $p_f \in M_n$. Then p_f is a best approximation to f from M_n if and only if there exist $x_i \in A$ $(i = 1,..., \mu)$ and $y_{ij} \in B_i$ $(j = 1,..., \lambda_i, i = 1,..., l)$ and positive numbers α_i $(i = 1,..., \mu)$ and β_{ij} $(j = 1,..., \lambda_i, i = 1,..., l)$ with*

$$\mu + \sum_{i=1}^{l} \lambda_i \leqslant n + 2 \tag{2.3}$$

such that

$$\sum_{i=1}^{\mu} \alpha_i \sigma(x_i)\, q(x_i) + \sum_{i=1}^{l} \epsilon_i \sum_{j=1}^{\lambda_i} \beta_{ij} q^{(k_i)}(y_{ij}) = 0 \tag{2.4}$$

for all $q \in \pi_n$.

In the remainder of this paper, p_f shall be the best approximation to f from M_n and the points x_i and y_{ij} shall be fixed. Let

$$\begin{aligned} \nu &= n \text{ if } \deg p_f \geqslant k_l \\ &= k_i - 1 \text{ if } k_i \text{ is the smallest } k_i > \deg p_f . \end{aligned}$$

Let e_i denote the number of y_{ij} in $\{a, b\}$ and $N = \mu - 1 + \sum_{k_i \leqslant \nu} (2\lambda_i - e_i)$. We shall be interested in the Birkoff interpolation problem (BIP) of finding a $q \in \pi_N$ such that

$$q(x_i) = a_i (i = 1,.,..\mu) \tag{2.5}$$

$$q^{(k_i)}(y_{ij}) = b_{ij} \qquad (j = 1,..., \lambda_i\, , k_i \leqslant \nu) \tag{2.6}$$

$$q^{(k_i+1)}(y_{ij}) = c_{ij} \qquad (a < y_{ij} < b, j = 1,..., \lambda_i\, , k_{i+1} \leqslant \nu). \tag{2.7}$$

The following lemma differs from Lemma 2.2 of R. A. Lorentz [6] in that he used $\deg p_f$ in place of ν an A and B_i in place of the x_i and y_{ij}. The proof is the same as Lorentz' proof with the exception of an application of (2.4) instead of another characterization theorem and is omitted.

LEMMA 2.2. *The* BIP (2.5)-(2.7) *consists of* $N + 1$ *nonoverlapping conditions and* $N \geqslant \nu + 1$. *The incidence matrix* E *for the* BIP (2.5)-(2.7) *satisfies the strong Polya condition and contains no odd supported sequences. Thus the* BIP (2.5)-(2.7) *has a unique solution in* π_N *for every choice of the* a_i, b_{ij}, *and* c_{ij}.

For reference on Birkoff interpolation, see Lorentz and Zeller [5].

For $p \in \pi_n$, define

$$\begin{aligned} \| p \|' = \max\{&| p(x_i)| \; (i = 1,..., \mu), \\ &| p^{(k_i)}(y_{ij})| \; (j = 1,..., \lambda_i\, , i = 1,..., l)\} \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \| p \|^* = \max\{&| p(x_i)| \; (i = 1,..., \mu), \\ &| p^{(k_i)}(y_{ij})| \; (j = 1,..., \lambda_i\, , i = 1,..., l), \\ &| p^{(k_i+1)}(y_{ij})| \; (a < y_{ij} < b, j = 1,..., \lambda_i\, , i = 1,..., l)\}. \end{aligned} \tag{2.9}$$

It is clear that $\|\cdot\|'$ and $\|\cdot\|^*$ are seminorms on π_n. The main use of Lemma 2.2 is in the next lemma.

LEMMA 2.3. *If* $\deg p_f \geqslant k_l$, *then* $\|\cdot\|^*$ *is a norm on* π_n.

Proof. The proof is easy. If $\deg p_f \geqslant k_l$, then $N > \nu = n$. As a result, the only solution to the homogeneous BIP (2.5)-(2.7)($a_i = b_{ij} = c_{ij} = 0$) in π_n is the trivial solution. If $p \in \pi_n$ and $\| p \|^* = 0$, then p is a solution of this homogeneous BIP and thus $p \equiv 0$.

An important consequence of Lemma 2.3 is that $\|\cdot\|^*$ and the uniform norm $\|\cdot\|$ taken as norms on the finite dimensional space π_n are equivalent.

3. STRONG UNICITY OF ORDER 1/2

We remark that if $f \in M_n$, then (1.1) holds with $\gamma = 1$. Henceforth, we shall assume that $f \notin M_n$.

The first theorem of this section is an extension of Theorem 4.2 of Fletcher and Roulier [2] to the more general setting of approximation from M_n $(k_1,..., k_l; \epsilon_1,..., \epsilon_l)$. The proof of this theorem is exactly the same as their proof and thus is omitted.

THEOREM 3.1. *Let* $f \in C[a, b]\backslash M_n$, p_f *be the best uniform approximation to* f *from* M_n, *and* $\|\cdot\|'$ *be given by* (2.8). *Then there is a constant* $\gamma > 0$ *such that*

$$\| f - p \| \geqslant \| f - p_f \| + \gamma \| p - p_f \|' \tag{3.1}$$

for all $p \in M_n$.

Fletcher and Roulier [2] proved independently of their Theorem 4.1 that (1.1) holds for those $p \in M_n$ for which $p_f - p \in M_n$. We give a simpler proof of this result using Theorem 3.1 and the ideas of Section 2. Again this result is extended to the more general monotone approximation problem.

THEOREM 3.2. *Let* $f \in C[a, b]\backslash M_n$ *and* p_f *be the best uniform approximation to* f *from* M_n. *If* $\deg p_f \geqslant k_l$, *then there is a constant* $\rho > 0$ *such that*

$$\| f - p \| \geqslant \| f - p_f \| + \rho \| p - p_f \|$$

for all $p \in M_n$ *with* $p^{(k_i)}(y_{ij}) = 0$ *for* $j = 1,..., \lambda_i$, $i = 1,..., l$.

Proof. Since $\|\cdot\|^*$ and $\|\cdot\|$ are equivalent norms on π_n, there is a constant $\rho_1 > 0$ such that $\| q \|^* \geqslant \rho_1 \| q \|$ for all $q \in \pi_n$. Let $p \in M_n$ with $p^{(k_i)}(y_{ij}) = 0$ for $j = 1,..., \lambda_i$, $i = 1,..., l$. Since $\epsilon_i p_f^{(k_i)}(x) \geqslant 0$ and $\epsilon_i p^{(k_i)}(x) \geqslant$

0 for $x \in [a, b]$, $p_f^{(k_i+1)}(y_{ij}) = p^{(k_i+1)}(y_{ij}) = 0$ for $a < y_{ij} < b$, $j = 1,..., \lambda_i$, $i = 1,..., l$. Hence, $\| p - p_f \|' = \| p - p_f \|^*$, and by (3.1) it follows that

$$\begin{aligned} \| f - p \| &\geqslant \| f - p_f \| + \lambda \| p - p_f \|^* \\ &\geqslant \| f - p_f \| + \lambda \rho_1 \| p - p_f \| . \end{aligned}$$

where λ depends only on f. The proof is completed by letting $\rho = \lambda \rho_1$.

The proof of the main result of this paper is similar to the proof of Theorem 3.2 in that we must obtain a relationship between $\| p - p_f \|'$ and $\| p - p_f \|^*$ for certain $p \in M_n$. In doing this we make use of the following lemma.

LEMMA 3.3. *Let* $p_m(x) = x^2 q_m(x) + \alpha_m x + \beta_m$ *for* $m = 1, 2,...,$ *where each* $q_m(x)$ *is a real-valued function defined on* $[-1, 1]$ *and the* α_m *and* β_m *are real. If there is a constant* $M > 0$ *such that* $q_m(x) \leqslant M$ *for all* $x \in [-1, 1]$ *and all* m, $p_m(x) \geqslant 0$ *for all* $x \in [-1, 1]$, *and* $\lim_{m \to \infty} \alpha_m = \lim_{m \to \infty} \beta_m = 0$, *then* $\alpha_m^2 \leqslant 4M\beta_m$ *for all sufficiently large* m.

Proof. Since $q_m(x) \leqslant M$ *for all* $x \in [-1, 1]$,

$$0 \leqslant p_m(x) \leqslant Mx^2 + \alpha_m x + \beta_m$$

for all $x \in [-1, 1]$. Assume $\alpha_m^2 > 4M\beta_m$ for infinitely many m. For such m, the quadratic function $Mx^2 + \alpha_m x + \beta_m$ has two distinct real zeros $(-\alpha_m \pm (\alpha_m^2 - 4M\beta_m))^{1/2}/2M$. Since $\lim_{m\to\infty} \alpha_m = \lim_{m\to\infty} \beta_m = 0$, $Mx^2 + \alpha_m x + \beta_m$ has two distinct real zeros in $(-1, 1)$ for infinitely many m. As a result, $Mx_m^2 + \alpha_m x_m + \beta_m < 0$ for some $x_m \in (-1, 1)$ and for infinitley many m. A contradiction is thus reached, and $\alpha_m^2 \leqslant 4M\beta_m$ for all sufficiently large m.

THEOREM 3.4. *Let* $f \in C[a, b] \backslash M_n$, *and let* p_f *be the best uniform approximation to* f *from* M_n. *If* $\deg p_f \geqslant k_l$, *then for each* $K > 0$ *there is a constant* $\gamma > 0$ *such that*

$$\| f - p \| \geqslant \| f - p_f \| + \gamma \| p - p_f \|^2 \tag{3.2}$$

for all $p \in M_n$ *with* $\| p \| \leqslant K$.

Proof. We assume that (3.2) does not hold. Then there is a sequence $\{p_m\}$ in M_n with $\| p_m \| \leqslant K$ and $\| p_m - p_f \| > 0$ such that

$$\gamma_m = \frac{\| f - p_m \| - \| f - p_f \|}{\| p_m - p_f \|^2} \to 0 \tag{3.3}$$

as $m \to \infty$. Since the $\| p_m \| \leqslant K$, we may assume that $p_m \to p \in M_n$ uniformly on $[a, b]$ as $m \to \infty$. Then by (3.3),

$$\begin{aligned}\| f - p \| &= \lim_{m\to\infty} \| f - p_m \| \\ &= \lim_{m\to\infty} (\| f - p_f \| + \gamma_m \| p_m - p_f \|^2) \\ &= \| f - p_f \|.\end{aligned}$$

Since best approximations from M_n are unique (see R. A. Lorentz [6]), $p = p_f$. Thus $p_m \to p_f$ uniformly on $[a, b]$ as $m \to \infty$.

Now by Theorem 3.1 there is a constant $\rho > 0$ such that

$$\| f - p \| \geqslant \| f - p_f \| + \rho \| p - p_f \|' \tag{3.4}$$

for all $p \in M_n$.

We now show that $(\| p_m - p_f \|^*)^2 \leqslant \rho_1 \| p_m - p_f \|'$ for some $\rho_1 > 0$ and all sufficiently large m. Since $\| \cdot \|^*$ and $\| \cdot \|$ are equivalent norms on π_n, $\| p_m - p_f \|^* \to 0$ as $m \to \infty$. Fix $y_{ij} \in (a, b)$. Since $p_f^{(k_i)}(y_{ij}) = p_f^{(k_i+1)}(y_{ij}) = 0$, $\lim_{m\to\infty} p^{(k_i)}(y_{ij}) = \lim_{m\to\infty} p^{(k_i+1)}(y_{ij}) = 0$. Also, $\epsilon_i p^{(k_i)}(x) \geqslant 0$ for all x in the neighborhood $[a, b]$ of y_{ij}. Since $\{p_m^{(k_i)}\}$ is uniformly convergent, we may write

$$p_m^{(k_i)}(x) = (x - y_{ij})^2 q_m(x) + p_m^{(k_i+1)}(y_{ij})(x - y_{ij}) + p_m^{(k_i)}(y_{ij})$$

where the $q_m(x)$ are uniformly bounded over $[a, b]$. Employing an appropriate linear change of variable and using Lemma 3.3, there is a constant $M_{ij} > 0$ such that

$$[p_m^{(k_i+1)}(y_{ij})]^2 \leqslant M_{ij} \mid p_m^{(k_i)}(y_{ij}) \mid$$

for all sufficiently large m. Thus

$$[p_m^{(k_i+1)}(y_{ij}) - p_f^{(k_i+1)}(y_{ij})]^2 \leqslant M_{ij} \mid p_m^{(k_i)}(y_{ij}) - p_f^{(k_i)}(y_{ij}) \mid$$

for m sufficiently large. Furthermore, $\| p_m - p_f \|' < 1$ for all sufficiently large m. Taking $\rho_1 = \max\{1, M_{ij}(a < y_{ij} < b, j = 1,..., \lambda_i, i = 1,..., l)\}$, we have

$$(\| p_m - p_f \|^*)^2 \leqslant \rho_1 \| p_m - p_f \|' \tag{3.5}$$

for m sufficiently large.

Since $\| \cdot \|^*$ and $\| \cdot \|$ are equivalent norms on π_n, there is a constant $\rho_2 > 0$ such that

$$\rho_2 \| p_m - p_f \| \leqslant \| p_m - p_f \|^* \tag{3.6}$$

for all m. By (3.4), (3.5), and (3.6), it now follows that

$$\|f - p_m\| \geqslant \|f - p_f\| + (\rho\rho_2/\rho_1)\|p_m - p_f\|^2$$

for all sufficiently large m. This contradicts (3.3) and Theorem 3.4 is proven.

We remark that the dependence of γ on K in Theorem 3.4 is essential. This is easy to see as $\|p - p_f\|^2$ grows faster than $\|f - p\|$ as $\|p\| \to \infty$. In addition, the example of Fletcher and Roulier [2] shows that the order $\frac{1}{2}$ cannot in general be improved. We briefly review their example. Let $[a, b] = [-1, 1]$, $n = 3$, $l = k_1 = \epsilon_1 = 1$, and

$$f(x) = \tfrac{1}{2} - x^2 + (x - 1/3^{1/2})^3.$$

Fletcher and Roulier showed that the best approximation to f from $M_3(1; 1)$ is

$$p_f(x) = (x - 1/3^{1/2})^3$$

and $\|f - p_f\| = \frac{1}{2}$. Moreover, for $p_\alpha \in M_3(1; 1)$ given by

$$p_\alpha(x) = (x - 1/3^{1/2})^2 + \alpha x(x^2 - (1 - \alpha)),$$

we have $\|f - p_\alpha\| = \frac{1}{2} + \alpha^2$ and $\|p_\alpha - p_f\| = 2\alpha(1 - \alpha)^{3/2}/3^{3/2}$ for α sufficiently small. For any $t < 2$,

$$\frac{\|f - p_\alpha\| - \|f - p_f\|}{\|p_\alpha - p_f\|^t} = \frac{\alpha^{2-t}}{[2(1 - \alpha)^{3/2}/3\sqrt{3}]^t} \to 0$$

as $\alpha \to 0$. Furthermore, $\|p_\alpha - p_f\| \to 0$ as $\alpha \to 0$. Hence p_f is not strongly unique of order greater than $\frac{1}{2}$.

4. Continuity and Lipschitz Conditions

The first theorem of this section asserts that the best monotone approximation operator is continuous with respect to the uniform norm topology on $C[a, b]$. As with Theorems 3.1 and 3.2, this theorem is an extension of a result of Fletcher and Roulier [2]. Unlike their proof, our proof does not ostensibly depend on Birkoff interpolation. Instead we employ the techniques of A. Kroó [3]. In this section, we shall let $T_n(f)$ denote the best approximation to f from M_n.

THEOREM 4.1. *The operator T_n is continuous at each $f \in C[a, b]$ with respect to the uniform norm topology on $C[a, b]$.*

Proof. Let $f \in C[a, b]$ and let $\{g_m\}$ be a sequence in $C[a, b]$ such that

$\lim_{m\to\infty} \| g_m - f \| = 0$. We must show that $\lim_{m\to\infty} \| T_n(g_m) - T_n(f) \| = 0$. Assume otherwise. Then by extracting a subsequence and relabeling, we may assume that $\| T_n(g_m) - T_n(f) \| \geqslant \epsilon$ for some $\epsilon > 0$ and all m. It can easily be seen that $\| T_n(g_m) \| \leqslant 2 \| g_m \|$ and hence the $T_n(g_m)$ are uniformly bounded. Thus we may assume that $T_n(g_m) \to q \in M_n$ uniformly on $[a, b]$ as $m \to \infty$. The inequality

$$\big| \| g_m - T_n(g_m) \| - \| f - T_n(f) \| \big| \leqslant \| g_m - f \|$$

appears in A. Kroó [3] and is easily established. Hence,

$$\lim_{m\to\infty} \| g_m - T_n(g_m) \| = \| f - T_n(f) \|.$$

But

$$\lim_{m\to\infty} \| g_m - T_n(g_m) \| = \| f - q \|.$$

By the uniqueness of best monotone approximations (see [6]), $q = T_n(f)$ and $\lim_{m\to\infty} \| T_n(g_m) - T_n(f) \| = 0$. This is a contradiction and the theorem is established.

The next theorem shows that if $\deg T_n(f) \geqslant k_l$, then the continuity of Theorem 4.1 is a local Lipschitz continuity of order $\frac{1}{2}$.

THEOREM 4.2. *Let $f \in C[a, b]$ and assume that* $\deg T_n(f) \geqslant k_l$. *Then for each $K > 0$ there is a constant $\lambda > 0$ such that*

$$\| T_n(g) - T_n(f) \| \leqslant \lambda \| g - f \|^{1/2} \tag{4.1}$$

for all $g \in C[a, b]$ with $\| g \| \leqslant K$.

Proof. The proof follows directly from Theorem 3.4. If $\| g \| \leqslant K$, then $\| T_n(g) \| \leqslant 2K$. Select $\gamma > 0$ such that

$$\| f - p \| \geqslant \| f - T_n(f) \| + \gamma \| p - T_n(f) \|^2$$

for all $p \in M_n$ with $\| p \| \leqslant 2K$. If $g \in C[a, b]$ and $\| g \| \leqslant K$, then

$$\begin{aligned} \| T_n(g) - T_n(f) \| &\leqslant \gamma^{-1/2} (\| f - T_n(g) \| - \| f - T_n(f) \|)^{1/2} \\ &\leqslant \gamma^{-1/2} (\| f - g \| + \| g - T_n(g) \| - \| f - T_n(f) \|)^{1/2} \\ &\leqslant \gamma^{-1/2} (\| f - g \| + \| g - T_n(f) \| - \| f - T_n(f) \|)^{1/2} \\ &\leqslant 2^{1/2} \gamma^{-1/2} \| g - f \|^{1/2}. \end{aligned}$$

Thus (4.1) holds with $\lambda = (2/\gamma)^{1/2}$.

5. Conclusions

The main results of this paper are Theorems 3.4 and 4.2. Although strong unicity of best monotone approximations can fail, strong unicity of order $\frac{1}{2}$ holds when $\deg p_f \geqslant k_l$. This condition is not overly stringent for if $f \notin M_{k_l-1}$, then $\deg p_f \geqslant k_l$ for all sufficiently large n. This condition is used only in proving that $\| \cdot \|^*$ is a norm on π_n. It would be interesting to investigate the necessity of this condition.

A subtle difference between Theorem 3.4 and ordinary strong unicity is that (3.2) holds on bounded subsets of M_n. This local nature cannot be avoided for strong unicity of order less than 1. This follows because $\| p - p_f \|^{1/\alpha}$ grows faster than $\| f - p \|$ as $\| p \| \to \infty$ when $\alpha < 1$.

Although the order $\frac{1}{2}$ is best possible for strong unicity, it is unknown whether the order $\frac{1}{2}$ is best possible for the Lipschitz condition.

It is the author's view that further research on strong uniqueness need be done for constrained approximation problems such as that of approximation with restricted ranges on derivatives (see [7]). This view is based on the observation that strong unicity appears in several convergence analyses for algorithms to compute best approximations.

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